CHAPTER 4

THE INTEGRAL

4.1 Summation

PREREQUISITES

1. There are no prerequisites other than simple addition and algebra for this section.

GOALS

1. Be able to manipulate expressions involving summation notation.

STUDY HINTS

1. Dummy index. Changing the index letter does not change the value of the sum. Understand this concept.
2. Properties of summation. With use, you will soon have these properties memorized, so there is no need to actively memorize these properties. However, you should understand the meaning of each statement. Note how $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ is analogous to $\int (f + g)dx = \int f\,dx + \int g\,dx$, which was introduced in Section 2.5 (See p. 130).
3. Sums of products. Be cautious that the following statement does not hold in general: $\sum_{i=m}^{n} a_i b_i \stackrel{?}{=} \left(\sum_{i=m}^{n} a_i \right) \left(\sum_{i=m}^{n} b_i \right)$.

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4. Sum of the first n numbers. Memorize or learn how to derive this formula.

5. Substitution of index. Formula (6) should be understood and not memorized. Note that the left-hand side of the formula in the text begins at a_{m+q} and ends at a_{n+q}. Be sure the sums begin and end with the same terms. For example, if $\sum_{j=1}^{5} 2^{j+2} = \sum_{i=x}^{y} 2^{i-3}$, then $x = 6$ and $y = 10$ because the exponents of 2 on both sides should begin at 3 and end at 7. (Note that the indices do not always start at 1 or 0.)

6. Telescoping sums. There are several other ways for writing the telescoping sum, such as $\sum_{i=m}^{n}[a_i - a_{i-1}] = a_n - a_{m-1}$. These formulas can easily be recognized by their minus signs and in many cases, a shift of the index by one. Again, don't memorize these formulas. To compute a telescoping sum, write down a few terms at the beginning and at the end of the sum, and then cancel out the terms in the middle.

Try Example 7 with $(i - 1)^3$ replaced by $(i + 3)^3$. This new example demonstrates that your answer may have more than two terms.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Use the formula $\Delta d = \sum_{i=1}^{n} v_i \Delta t_i$ to get $\Delta d = 2(3) + 1.8(2) + 2.1(3) + 3(1.5) = 20.4$ meters.

5. $\sum_{i=1}^{4} (i^2 + 1) = (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) = 2 + 5 + 10 + 17 = 34$.

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9. Use the formula for summing the first n integers to get $1 + 2 + \ldots +$ $25 = \sum_{i=1}^{25} i = (1/2)(25)(26) = 325$.

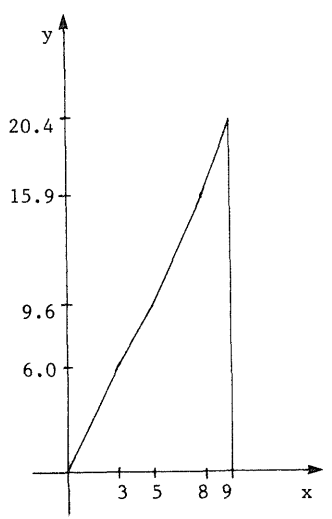
13. Use the method of Example 6 and then use the formula for the sum of the first n integers to get $\sum_{j=4}^{80} (j - 3) = \sum_{i=1}^{77} i = (1/2)(77)(78) = 3003$.

17. Note that the n^{th} term cancels the $-(n^{th})$ term, so $\sum_{j=-2}^{2} j^3 = 0$.

21. Using property 3 of summation, we have $\sum_{i=1}^{n} i = \sum_{i=1}^{m-1} i + \sum_{i=m}^{n} i$. Rearrange the equation and use the formula for the sum of the first n integers to get $\sum_{i=m}^{n} i = \sum_{i=1}^{n} i - \sum_{i=1}^{m} i = (1/2)(n)(n + 1) - (1/2)(m - 1)(m)$.

25. Use the method of Example 7 to compute the telescoping sum. $\sum_{i=1}^{100} [i^4 - (i - 1)^4] = [(1)^4 - (0)^4] + [(2)^4 - (1)^4] + \ldots + [(99)^4 - (98)^4] + [(100)^4 - (99)^4] = (100)^4 - (0)^4 = 100{,}000{,}000$.

29. Graphically, the velocity is the slope.



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33. This telescoping sum is $\sum_{k=1}^{n} [(k+1)^4 - k^4] = [(2)^4 - (1)^4] +$ $[(3)^4 - (2)^4] + \ldots + [(n-1+1)^4 - (n-1)^4] + [(n+1)^4 - n^4] =$ $(n+1)^4 - 1$.

37. $\sum_{i=-30}^{30} (i^5 + i + 2) = \sum_{i=-30}^{30} (i^5 + i) + \sum_{i=-30}^{30} 2$. Notice that the n^{th} term cancels the $-(n^{th})$ term in $\sum_{i=-30}^{30} (i^5 + i)$. Using Property 4 of summation, the sum is $0 + 2(61) = 122$.

41. (a) Rearrange the given equation and sum both sides. $3 \sum_{i=1}^{n} i^2 =$ $\sum_{i=1}^{n} [(i+1)^3 - i^3] - 3 \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = [(n+1)^3 - 1] -$ $3[n(n+1)/2] - n = [n^3 + 3n^2 + 3n] - [(3n^2 + 3n)/2] - n = n^3 +$ $(3/2)n^2 + (1/2)n$. Dividing both sides by 3 , we get

$$\sum_{i=1}^{n} i^2 = n^3/3 + n^2/2 + n/6 = [n(n+1)(2n+1)]/6 .$$

(b) If m is positive, then $\sum_{i=m}^{n} i^2 = \sum_{i=0}^{n} i^2 - \sum_{i=0}^{m-1} i^2 =$ $[n(n+1)(2n+1)]/6 - [(m-1)m(2m-1)]/6$.

If m is 0 , then $\sum_{i=0}^{n} i^2 = [n(n+1)(2n+1)]/6$ and the previous result still holds.

If m is negative, the $\sum_{i=m}^{n} i^2 = \sum_{i=0}^{n} i^2 + \sum_{i=m}^{0} i^2 =$ $[n(n+1)(2n+1)]/6 + [m(m+1)(2m+1)]/6$. However, we get the same result by substituting $-m$ for m in $[n(n+1)(2n+1)]/6 -$ $[(m-1)m(2m-1)]/6$; therefore, $\sum_{i=m}^{n} i^2 = [n(n+1)(2n+1)]/6 -$

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41. (b) continued.

$[(m - 1)m(2m - 1)]/6 = (1/6)[2(n^3 - m^3) + 3(n^2 + m^2) + (n - m)]$,

regardless of the sign of m.

(c) $\sum_{i=1}^{n} [(i + 1)^4 - i^4] = (n + 1)^4 - 1$ because this is a telescoping sum. $(i + 1)^4 - i^4 = 4i^3 + 6i^2 + 4i + 1$. Rearrangement and summing yields $4 \sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} [(i + 1)^4 - i^4] - 6 \sum_{i=1}^{n} i^2 - 4 \sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = [(n + 1)^4 - 1] - [n(n + 1)(2n + 1)] - 2n(n + 1) - n = n^2(n^2 + 2n + 1)$. Thus, $\sum_{i=1}^{n} i^3 = [n(n + 1)/2]^2$.

SECTION QUIZ

1. Compute $\sum_{i=1}^{n} 2 + \sum_{i=-5}^{m} i + \sum_{j=0}^{n} 3j$. Assume that m and n are positive integers.

2. Find x and y, and then compute the resulting sum: $\sum_{j=-4}^{45} [(j + 6)^2 - (j + 4)^2] = \sum_{k=x}^{y} [(k + 2)^2 - k^2]$.

3. Find $\sum_{i=0}^{3} 3i$ and $\left(\sum_{i=0}^{3} 3\right)\left(\sum_{i=0}^{3} i\right)$. What fact does your answer demonstrate?

4. (a) Consider $\sum_{i=2}^{2} i$. Is this expression defined? If so, what is its value?

   (b) Consider $\sum_{i=2}^{1} i$. Is this expression defined? If so, what is its value?

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5. Your rich, eccentric uncle's will requests that you drive his ashes around town in his Rolls Royce. He offers to pay you $7000j$ dollars on the j^{th} day of the month. On the other hand, he will pay you 2^{j+1} cents for this service on the j^{th} day provided you return 2^{j-1} cents. If you work during April, how much could you earn by each form of payment? (Note: $2^{30} = 1{,}073{,}741{,}824$.)

ANSWERS TO SECTION QUIZ

1. $2n + m(m + 1)/2 - 15 + 3n(n + 1)/2$.

2. $x = 0$, $y = 49$; sum $= (51)^2 + (50)^2 - (1)^2 - (0)^2 = 5100$.

3. $\sum_{i=0}^{3} 3i = 18$; $\left(\sum_{i=0}^{3} 3\right)\left(\sum_{i=0}^{3} i\right) = 12(6) = 72$. This shows that the sum of products does not equal the product of the sums of the multiplicands.

4. (a) Yes. $\sum_{i=2}^{2} i = 2$.
   (b) No.

5. \$3,255,000 by the first method; \$32,212,254.69 by the second method.

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4.2 Sums and Areas

PREREQUISITES

1. Recall how to manipulate expressions involving the summation notation (Section 4.1).

PREREQUISITE QUIZ

1. Fill in the blank: $\sum_{i=1}^{n} (i + 3) =$ _____ $+ \sum_{i=1}^{n} i$.

2. Compute the sum in Question 1 .

3. Consider $\sum_{j=1}^{4} x_j y_j$. Let $x_j = 2^j$ and $y_j = j - 2$. Compute the sum.

GOALS

1. Be able to find x_i , Δx_i , and k_i for a given step function, and be able to find the area of the region under its graph.
2. Be able to state the relationship between upper sums, lower sums, and the area under a positive function.

STUDY HINTS

1. Area under graphs. The area under a graph is closely related to the important concept of the integral, so be sure you know exactly what the boundaries of the area are. Note that one or both of a and b in Fig. 4.2.2 could be negative.
2. Step functions. Be able to define a step function. In addition, be familiar with the notations x_i , Δx_i , and k_i . Note that the first x is x_0 , not x_1 . This is because there are $n + 1$ partition points and only n intervals.

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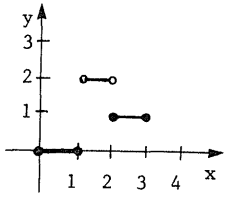
3. Upper and lower sums. Note that a lower sum is the area of any step function which lies entirely within the region under a graph. Thus, there are infinitely many lower sums. Also, note that the subintervals may have different widths. Similar statements may be made for upper sums.

4. Relationship between upper sums, lower sums, and area. Know the following inequality: lower sums $\leq$ area under a graph $\leq$ upper sums.

5. Direct calculation of areas. Example 6 shows how areas were computed prior to the invention of calculus. Do not be overly concerned with understanding this example. You will be given a simple formula for computing areas in Section 4.4 .

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

As in Example 1, the graph consists of horizontal lines with heights 0 , 2 and 1 on their respective intervals. Solid dots are used to indicate that the function includes the endpoint. Open dots indicate that the endpoint is not to be included.

5. x_0 is the first endpoint. The other x_i's occur where the function changes value or at the last endpoint. Thus, $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. By definition, $\Delta x_i = x_i - x_{i-1}$; so $\Delta x_1 = x_1 - x_0 = 1 - 0 = 1$. Similarly, $\Delta x_2 = \Delta x_3 = 1$. The k_i's are the function values on the i^{th} interval, so $k_1 = 0$, $k_2 = 2$, and $k_3 = 1$.

The area under the graph is $\sum_{i=1}^{3} k_i \Delta x_i = 0(1) + 2(1) + 1(1) = 3$.

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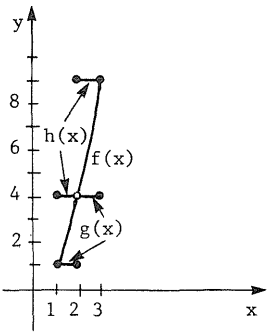
9. 

The upper sum is the area under $h(x)$, shown as a dotted line. Its area is $\sum_{j=1}^{2} \ell_j \Delta x_j = 4(1) + 9(1) = 13$. The lower sum is the area under $g(x)$, shown as horizontal solid lines. The area is $\sum_{i=1}^{2} k_i \Delta x_i = 1(1) + 4(1) = 5$.

13. The problem is analogous to Example 5. According to Exercise 9, the lower and upper sums are 5 and 13, so the distance crawled is between 0.010 and 0.026 meters, i.e., between 10 and 26 millimeters.

17. Using the method of Example 6, partition $[a,b]$ into n equal subintervals. Thus, the partition is $(a, a + (b - a)/n, a + 2(b - a)/n, \ldots, a + (n - 1)(b - a)/n, b)$. We will find upper and lower sums and use the fact that $L \leq A \leq U$. On the interval $(a + (i - 1)(b - a)/n, a + i(b - a)/n)$, let $g(x) = 5[a + (i - 1)(b - a)/n]$ and let $h(x) = 5[a + i(b - a)/n]$. Therefore, a lower sum is $\sum_{i=1}^{n} 5[a + (i - 1)(b - a)/n][(b - a)/n]$ because $\Delta x_i = (b - a)/n$. This is $[5a(b - a)/n] \sum_{i=1}^{n} 1 + [5(b - a)^2/n^2] \sum_{i=0}^{n-1} i = 5a(b - a) + 5(b - a)^2(n - 1)/2n = 5(ab - a^2) + 5(b^2 - 2ab + a^2)/2 - 5(b - a)^2/2n = 5(b^2 - a^2)/2 - 5(b - a)^2/2n$.

Similarly, an upper sum is $\sum_{i=1}^{n} 5[a + i(b - a)/n][(b - a)/n] = [5a(b - a)/n] \sum_{i=1}^{n} 1 + [5(b - a)^2/n^2] \sum_{i=1}^{n} i = 5a(b - a) + 5(b - a)^2(n + 1)/2n = 5(b^2 - a^2)/2 + 5(b - a)^2/2n$. Since $L \leq A \leq U$ for all n, we must have $A = 5(b^2 - a^2)/2$.

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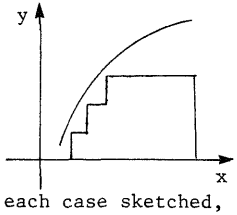
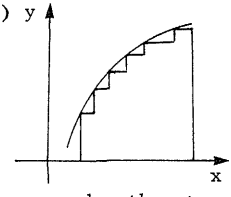
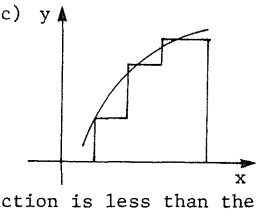
21. According to the result of Exercise 15, the area x on $[1,2]$ is $3/2$. By the result of Exercise 20, the area under x^2 on $[0,1)$ is $(1/3)(1^3 - 0^3) = 1/3$. Finally, by the additive property for areas, the area under $f(x)$ on $[0,2]$ is $3/2 + 1/3 = 11/6$.

SECTION QUIZ

1. True or false: For a given partition, x_0 may be negative.
2. True or false: If you know Δx_1, then all of the other Δx_i must equal Δx_1.
3. For a given non-negative function, can a lower sum ever equal an upper sum? Explain your answer.
4. (a) 

(b) 

(c) 

In each case sketched, the area under the step function is less than the area under $f(x)$. Which, if any, of the step functions is a lower sum?
5. Your new office building was constructed by a carpenter who enjoyed cocktail lunches. Upon completion, you discover that the drunk has made one of the walls curved. Measured in feet, the wall follows the curve $y = x^3 + 6x^2 + 3$ on $[-5,3]$.
   (a) The floor is to be lined with one-foot square tiles which cannot be cut. The sides of the tiles are placed parallel to the coordinate axes. How much floor area may be tiled?
   (b) Upon closer inspection, the curved wall has a crack on the bottom which permits tile to be slipped under the wall. How many tiles are needed to completely tile the floor?

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5. (c) From your answers in parts (a) and (b), what do you know about the exact area of the floor?

ANSWERS TO PREREQUISITE QUIZ

1. $\sum_{i=1}^{n} 3 = 3n$

2. $n^2/2 + 7n/2$

3. 38

ANSWERS TO SECTION QUIZ

1. True.

2. False. Δx_i may have any positive length.

3. They are equal only when the number is precisely the area under the curve.

4. (a) and (b). (c) does not satisfy $g(x) \leqslant f(x)$ on $[a,b]$.

5. (a) 136 square feet.

   (b) 256 tiles.

   (c) 136 sq. ft. $\leqslant$ area $\leqslant$ 256 sq. ft.

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4.3 Definition of the Integral

PREREQUISITES

1. Recall how step functions are used to determine the area under a positive function (Section 4.2).
2. Recall the definitions for upper and lower sums (Section 4.2).

PREREQUISITE QUIZ

1. Define upper and lower sums. Give summation formulas for computing them.
2. How are upper and lower sums related to the area under the graph of a positive function?

GOALS

1. Be able to use the concepts of step functions and upper and lower sums from Section 4.2 to estimate the signed area under a general function.
2. Be able to relate signed area to the integral.
3. Be able to write integrals as a Riemann sum.

STUDY HINTS

1. Signed area. Know that signed area means that the area below the x-axis is subtracted from the area above the x-axis. The computational formula is the same as the one used for area in the last section.
2. Upper sum, lower sum, and area. The definitions and the formulas for upper and lower sums are the same as those in Section 4.2 . Also, upper and lower sums are related to signed areas in exactly the same way they were related to areas in the previous section.

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3. Integrals and signed areas. It is important to know that the integral and the signed area under a curve are equal.

4. Definition of the integral. We will define the integral to be the unique number which lies between all lower sums and all upper sums.

5. Estimating integrals. Referring to Example 4, note that estimates usually use equal subdivisions. In general, more subdivisions increase the accuracy of the estimate. Also note that for functions which are strictly increasing or decreasing, your upper and lower sums will differ only in the first and last terms. This is because the estimate uses a telescoping sum. Thus, the difference is $|f(x_0) - f(x_n)|\Delta x$. When estimating integrals of general functions, be sure to consider the critical points when you compute upper and lower sums. Why consider the critical points? Try to estimate a lower sum for $f(x) = x^2 - x$ on $[0,1]$ with $n = 1$.

6. Integrability versus differentiability. Non-continuous functions may be integrated whereas they cannot be differentiated. For example, a step function may be integrated over an interval, but it is not differentiable.

7. Riemann sums. Note that c_i may be chosen as any point in $[x_{i-1},x_i]$. Note also that the definition requires the number of subdivisions to go to ∞ and that the largest subdivision approach zero.

8. Physical motivation. Although the supplement may be skipped, it will give you an appreciation of the usefulness and the practicality of the integral.

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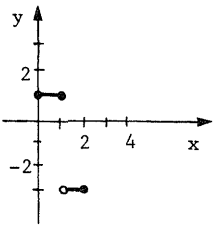
SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. 

The signed area of the step function depicted at the left is $\sum_{i=1}^{2} k_i \Delta x_i = (1)(1 - 0) + (-3)(2 - 1) = -2$.

5. $\int_a^b g(x)dx$ is the signed area of the region between $g(x)$ and the x-axis from $x = a$ to $x = b$. From the solution to Exercise 1, we know that $\int_0^2 g(x)dx = -2$.

9. Since the signed area equals the integral, the answer is $\int_{-1}^{2} x^2 dx$.

13. Applying the method of Example 4, we use the partition $(2,8/3,10/3,4)$. Thus, an upper sum is $[1/2 + 1/(8/3) + 1/(10/3)](2/3) = 37/60$. Similarly, a lower sum is $[1/(8/3) + 1/(10/3) + 1/4](2/3) = 47/60$. The integral lies between $37/60$ and $47/60$, a difference of $1/6$. Therefore, $42/60 = 7/10$ must be within $1/12$ of $\int_2^4 (dx/x)$. A better estimate may be obtained by using smaller subdivisions.

17. This problem is analogous to Example 5.

(a) The displacement of the bus is $\int_0^3 5(t^2 - 5t + 6)dt$.

(b) Note that $v = 5(t - 3)(t - 2)$, so it is positive on $(0,2)$ and negative on $(2,3)$. Thus, the total distance travelled is $\int_0^2 5(t^2 - 5t + 6)dt - \int_2^3 5(t^2 - 5t + 6)dt$.

21. Divide the interval into n equal parts to get the partition $(2 , 2 + 2/n , 2 + 4/n , \ldots , 2 + 2(n - 1)/n , 4)$. Choose $c_i = 2 + 2i/n$, so $S_n = \sum_{i=1}^{n} \{1/[1 + (2 + 2i/n)^2]\}(2/n) = \sum_{i=1}^{n} [n^2/(n^2 + 4n^2 + 8in + 4i^2)](2/n) = \sum_{i=1}^{n} [2n/(n^2 + 4n^2 + 8in + 4i^2)]$. Therefore,

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21. (continued)

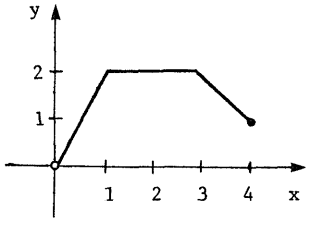
$\lim_{n\to\infty} \sum_{i=1}^{n} [2n/(5n^2 + 8in + 4i^2)] = \int_2^4 [dx/(1 + x^2)]$. (Answers may vary depending on the choice of c_i.)

25. (a) If $0 \leq x < 1$, we can take $(0,x)$ as our partition and $\int_0^x f(t)dt = 2(x - 0) = 2x$. If $1 \leq x < 3$, we take $(0,1,x)$ as our partition and $\int_0^x f(t)dt = 2\cdot(1 - 0) + 0\cdot(x - 1) = 2$. If $3 \leq x \leq 4$, we take $(0,1,3,x)$ as our partition and $\int_0^x f(t)dt = 2(1 - 0) + 0(3 - 1) + (-1)(x - 3) = 2 - x + 3 = -x + 5$.

Summarizing, we have

$$\int_0^x f(t)dt = \begin{cases} 2x & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 1 \leq x < 3 \\ -x + 5 & \text{if } 3 \leq x \leq 4 \end{cases} .$$

(b)



(c) We can see from the graph that F is differentiable everywhere on $(0,4]$ except at the points 1 , 3 , and 4 . We have:

$$F'(x) = \begin{cases} 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } 1 < x < 3 \\ -1 & \text{if } 3 < x < 4 \end{cases} .$$

We see that F' is the same as f , except at the points where the piecewise constant function f has a jump.

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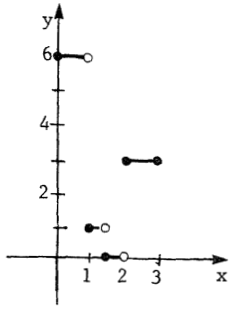
29. (a) The total volume is the sum of the volumes of the individual pieces, i.e., $\sum_{i=1}^{n} A_i \Delta x_i$.

(b) Consider a graph of cross-sectional area, $f(x)$, versus the distance, x , from the end of the rod. The volume is the area under the curve or $\int_0^L f(x)dx$ (assuming f to be integrable).

33. (a)



$$f(x) + g(x) = \begin{cases} 6 & 0 \leq x < 1 \\ 1 & 1 \leq x < 3/2 \\ 0 & 3/2 \leq x < 2 \\ 3 & 2 \leq x \leq 3 \end{cases} .$$

This is a step function. Use the partition $(0 , 1 , 3/2 , 2 , 3)$. Therefore, $\int_0^3 [f(x) + g(x)]\,dx =$ $6(1 - 0) + 1(3/2 - 1) + 0(2 - 3/2) + 3(3 - 2) = 19/2$.

(b) Use the partition, $(1 , 3/2 , 2)$, so $\int_1^2 [f(x) + g(x)]\,dx = 1(3/2 - 1) + 0(2 - 3/2) = 1/2$.

(c) $\int_0^3 2f(x)dx = 12(1 - 0) + 2(3/2 - 1) + 0(2 - 3/2) + 6(3 - 2) = 19$; $2\int_0^3 f(x)dx = 2(19/2) = 19 = \int_0^3 2f(x)dx$.

(d) $$f(x) - g(x) = \begin{cases} 2 & 0 \leq x < 1 \\ -3 & 1 \leq x < 3/2 \\ -2 & 3/2 \leq x < 2 \\ 1 & 2 \leq x \leq 3 \end{cases} .$$

Using the same partition as in part (a), $\int_0^3 [f(x) - g(x)]\,dx =$ $2(1 - 0) + (-3)(3/2 - 1) + (-2)(2 - 3/2) + 1(3 - 2) = 1/2$.

(Continued on next page.)

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33. (d) (continued)

$\int_0^3 f(x)dx - \int_0^3 g(x)dx = [(4)(1 - 0) + (-1)(3/2 - 1) + (-1)(2 - 3/2) + (2)(3 - 2)] - [(2)(1 - 0) + (2)(3/2 - 1) + (1)(2 - 3/2) + (1)(3 - 2)] = 5 - 9/2 = 1/2$. Therefore, $\int_0^3 [f(x) - g(x)]dx = \int_0^3 f(x)dx - \int_0^3 g(x)dx$.

(e) $$f(x)\cdot g(x) = \begin{cases} 8 & 0 \leq x < 1 \\ -2 & 1 \leq x < 3/2 \\ -1 & 3/2 \leq x < 2 \\ 2 & 2 \leq x \leq 3 \end{cases} .$$

$\int_0^3 f(x)\cdot g(x)dx = 8(1 - 0) + (-2)(3/2 - 1) + (-1)(2 - 3/2) + 2(3 - 2) = 15/2$. From part (d), we have $\int_0^3 f(x)dx \cdot \int_0^3 g(x)dx = 5(9/2) = 45/2$; therefore, the statement is false.

SECTION QUIZ

1. State whether the following statements are true or false.
   (a) There is no such thing as a negative area.
   (b) Any integrable function is differentiable.
   (c) Any differentiable function is integrable.
2. Circle the appropriate word: The integral is (exactly, approximately, not) equal to the signed area under the curve.
3. You have purchased a strip of land from the Rolling Hills Real Estate Company. Along a straight line, the land has altitude $x^3 + 2x^2 - 4x - 3$, where the x-axis represents sea level. Measurements were made in kilometers. Your pet buffaloes like to roam over level ground, so soil from the hills is used to fill in the valleys.

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3. (a) Using the partition $(-3, -2, -1, 0, 1, 2, 3)$, find an upper sum and a lower sum to estimate the height of the ground on the interval $[-3,3]$.

   (b) Write the exact height as a limit.

ANSWERS TO PREREQUISITE QUIZ

1. An upper sum for a function $f(x)$ is the area under a step function $h(x)$ for which $h(x) \geqslant f(x)$ on the interval of definition. A lower sum is the area under a step function $g(x)$ for which $g(x) \leqslant f(x)$.

   The formulas are $U = \sum_{j=1}^{m} \ell_j \Delta x_j$ and $L = \sum_{i=1}^{n} k_i \Delta x_i$.

2. Lower sums $\leqslant$ Area $\leqslant$ Upper sums.

ANSWERS TO SECTION QUIZ

1. (a) False. This involves the concept of signed areas.

   (b) False. Consider the partition points of a step function.

   (c) True. All differentiable functions are continuous.

2. Exactly.

3. (a) Since the strip of land is 6 kilometers long, the upper sum is $44/6 = 22/3$ kilometers; Lower sum $= (-256/27)/6 = -128/51$ kilometers. (Did you consider the critical point at $x = 2/3$?)

   (b) $\lim_{n \to \infty} \left[(396/n^2) \sum_{i=1}^{n} i - (1452/n^3) \sum_{i=1}^{n} i^2 + (1296/n^4) \sum_{i=1}^{n} i^3\right]$.

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4.4 The Fundamental Theorem of Calculus

PREREQUISITES

1. Recall how to compute antiderivatives using the power rule, the sum rule, and the constant multiple rule (Section 1.4).

PREREQUISITE QUIZ

1. Find an antiderivative for the following functions:
   (a) x
   (b) x^4
   (c) x^n , n = integer, $\neq -1$.
2. Use the constant multiple rule to find an antiderivative for:
   (a) $3x$
   (b) $49x^6$
   (c) $(n + 1)x^n/2$, where n = integer, $\neq -1$
   (d) 5
3. Find an antiderivative for:
   (a) $x + 3$
   (b) $x^4 + 3x + 5$

GOALS

1. Be able to state and apply the fundamental theorem of calculus.
2. Be able to evaluate integrals by using antiderivatives.

STUDY HINTS

1. Fundamental theorem of calculus. $\int_a^b F'(x)dx = \int_a^b f(x)dx = F(b) - F(a)$ is one of the most important equations you will encounter in your study of mathematics. Be sure you understand how to use it.

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2. Notation. Get familiar with $F(x)\Big|_a^b$. It will be seen throughout the course.

3. Proof of the fundamental theorem. The proof depends on the concepts of upper sums, lower sums, telescoping sums, and the mean value theorem. Knowing how to use the theorem is of more immediate importance than understanding the proof.

4. Dummy variable. Changing the variable letter does not change the value of an integral. $\int_a^b f(N)dN$ is the same as $\int_a^b f(x)dx$.

5. Algebraic manipulations of integrands. Example 7 demonstrates how "hard" integrals may be converted to "easy" integrals. Expansion and division are common simplification techniques. Other methods will be introduced in subsequent chapters.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. By the power rule for antidifferentiation, an antiderivative for x^3 is $x^4/4$. Thus $\int_1^3 x^3 dx = (x^4/4)\Big|_1^3 = (3)^4/4 - (1)^4/4 = (81 - 1)/4 = 20$.

5. $F(x)\Big|_a^b = F(b) - F(a)$; therefore $x^{3/4}\Big|_0^2 = (2)^{3/4} - (0)^{3/4} = 8^{1/4}$.

9. $\int_a^b s^{4/3} ds = (3s^{7/3}/7)\Big|_a^b = (3/7)(b^{7/3} - a^{7/3})$. This exercise demonstrates the concept of a dummy variable.

13. The fundamental theorem yields $\int_0^{10}(t^4/100 - t^2)dt = (t^5/500 - t^3/3)\Big|_0^{10} = 100000/500 - 1000/3 = -400/3$.

17. From Section 2.5, we get the formula $\int(at + b)^n dt = (at + b)^{n+1}/a(n + 1) + C$. Thus, $\int_1^2[dt/(t + 4)^3] = \int_1^2(t + 4)^{-3}dt = [(t + 4)^{-2}/(-2)]\Big|_1^2 = -1/72 - (-1/50) = 11/1800$.

21. Expand and then perform division on the integrand to get $\int_1^2[(x^2 + 5)^2/x^4]dx = \int_1^2[(x^4 + 10x^2 + 25)/x^4]dx = \int_1^2 (1 + 10x^{-2} + 25x^{-4})dx = (x - 10x^{-1} - 25/3x^3)\Big|_1^2 = 3 - 10(-1/2) - (25/3)(-7/8) = 367/24$.

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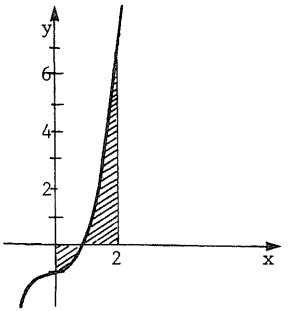
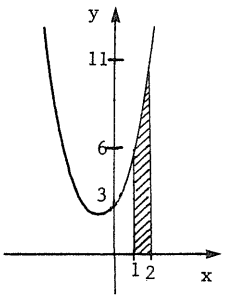
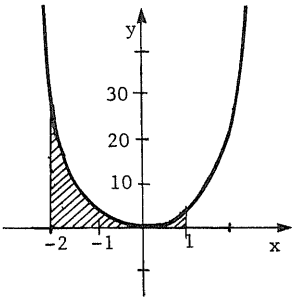
25. The region is that under the graph of $y = x^2$ from $x = 1$ to $x = 2$, so its area is $\int_1^2 x^2 dx$. By the fundamental theorem, $\int_1^2 x^2 dx = (x^3/3)\Big|_1^2 = 7/3$. Thus, the area is $7/3$.

29. 

An antiderivative for $x^3 - 1$ is $x^4/4 - x$, so $\int_0^2 (x^3 - 1)dx = (x^4/4 - x)\Big|_0^2 = 16/4 - 2 = 2$. The integral represents the signed area between $x^3 - 1$ and the x-axis from $x = 0$ to $x = 2$, as depicted by the shaded region at the left.

33. 

The graph of $x^2 + 2x + 3$ lies above the x-axis, so the area between the function and the x-axis in the interval $[1,2]$ is given by $\int_1^2 (x^2 + 2x + 3)dx = (x^3/3 + x^2 + 3x)\Big|_1^2 = 7/3 + 3 + 3 = 25/3$.

37. 

The graph of $x^4 + 3x^2 + 1$ lies above the x-axis, so the area between the function and the x-axis in the interval $[-2,1]$ is given by $\int_{-2}^1 (x^4 + 3x^2 + 1)dx = (x^5/5 + x^3 + x)\Big|_{-2}^1 = 33/5 + 9 + 3 = 93/5$.

41. The displacement equals the total distance travelled since $6t^4 + 3t^2 > 0$. Thus, the displacement is $\Delta d = \int_1^{10} (6t^4 + 3t^2)dt = (6t^5/5 + t^3)\Big|_1^{10} = 121{,}000 - 11/5 = (604{,}989/5)$ units.

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45. The distance travelled is the integral of the velocity, which is $\int_0^{10} 32t\,dt = 16t^2\Big|_0^{10} = 1600$ feet.

49. This exercise completes the proof of the fundamental theorem. Choose an appropriate partition $(x_0, x_1, x_2, \ldots, x_m)$ and let $\ell_1, \ell_2, \ldots, \ell_m$ be the values of h on the partition intervals. On (x_{i-1}, x_i), $F'(t) = f(t) \leq \ell_i$. By Consequence 1 of the mean value theorem $[F(x_i) - F(x_{i-1})]/[x_i - x_{i-1}] \leq \ell_i$, or $[F(x_i) - F(x_{i-1})] \leq \ell_i \Delta x_i$. Summing for $i = 1$ to m yields a telescoping sum on the left, while the right-hand side is the integral of h on $[a,b]$. Therefore, $F(b) - F(a) \leq \int_a^b h(t)\,dt$.

SECTION QUIZ

1. State the fundamental theorem of calculus.

2. Compute:

   (a) $\int_{-3}^{3} x\,dx$

   (b) $\int_{-2}^{2} x^3\,dx$

   (c) $\int_{-a}^{a} x^n\,dx$, where a is a positive integer and n is an odd, non-negative integer.

3. State a theorem using the result of Question 2(c).

4. Is $\int_{-8/9}^{100/17} (x^7 + 6x^6 + 4x^3 + 53)\,dx$ equal to $\int_{-8/9}^{100/17} (A^7 + 6A^6 + 4A^3 + 53)\,dA$?

5. Compute $\int_0^1 x(x + 1)(x + 2)\,dx$.

6. The bottom of the swimming pool outside your mansion is bounded by $y = 3 - 2x - x^2$, the lines $x = -1$ and $x = 2$, and the x-axis. It is to be lined with gold. Since gold is so expensive, you need to find the exact area. What is it?

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ANSWERS TO PREREQUISITE QUIZ

1. (a) $x^2/2 + C$

   (b) $x^5/5 + C$

   (c) $x^{n+1}/(n + 1) + C$

2. (a) $3x^2/2 + C$

   (b) $7x^7 + C$

   (c) $x^{n+1}/2 + C$

   (d) $5x + C$

3. (a) $x^2/2 + 3x + C$

   (b) $x^5/5 + 3x^2/2 + 5x + C$

ANSWERS TO SECTION QUIZ

1. $\int_a^b F'(x)dx = F(b) - F(a)$

2. (a) 0

   (b) 0

   (c) 0

3. If n is an odd integer, then $\int_{-a}^{a} x^n dx = 0$. There are no restrictions on a .

4. Yes, only the dummy variable has changed.

5. $\int_0^1 x(x + 1)(x + 2)dx = \int_0^1 (x^3 + 3x^2 + 2x)dx = 9/4$.

6. $\int_{-1}^{2} (3 - 2x - x^2)dx = 3$ square units.

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4.5 Definite and Indefinite Integrals

PREREQUISITES

1. Recall the rules for differentiating sums, products, quotients, and a power of a function (Section 1.3).
2. Recall the properties of summation (Section 4.1).
3. Recall the fundamental theorem of calculus (Section 4.4).

PREREQUISITE QUIZ

1. Differentiate the following functions:
   (a) $x^2 + 3x + 1$
   (b) $(x^5 + x)(x^2 + 3x + 1)$
   (c) $(x + 3)/(x^2 + 1)$
   (d) $(x + 10)^3/x$
2. Fill in the blank.
   (a) $\sum_{i=1}^{25} (i^2 + i + 1) = \sum_{i=1}^{25} (i + 1) +$ _____ .
   (b) $\sum_{i=1}^{7} (6/i) =$ _____ $\times \sum_{i=1}^{7} (1/i)$.
   (c) $\sum_{k=3}^{10} k = \sum_{k=3}^{5} k +$ _____ .
3. Compute $\sum_{j=0}^{15} 4$.
4. What is the relationship between the integral and the derivative?

GOALS

1. Be able to state the alternative version of the fundamental theorem of calculus.
2. Be able to use the properties of integration to compute integrals.
3. Be able to check any integration formula by differentiating.

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STUDY HINTS

1. Definite vs. indefinite integrals. The box on p. 232 relates the definite integral, $\int_a^b f(x)dx$, on the right-hand side to the indefinite integral, $\int f(x)dx$, on the left. The box simply shows another way to express the fundamental theorem of calculus. You should recognize that definite integrals specify their endpoints and possess a unique value. Remember that indefinite integrals always have an additive constant.

2. Indefinite integral test. The value of this test cannot be emphasized enough. Since differentiation is much easier than integration, it is a good habit to check your integrals by differentiating.

3. Properties of integration. You should become familiar with the statements made in the box on p. 234. As with the properties of summation, the properties of integration will be memorized with use. Notice how Properties 1 and 2 are similar to the differentiation rules with the same name.

4. Products of integrals. In general, the integral of a product is not the product of each multiplicand's integral, i.e., $\int_a^b f(x)g(x)dx \neq \int_a^b f(x)dx \cdot \int_a^b g(x)dx$. The technique for integrating products will be shown in Section 7.4 .

5. Proof of the properties. Your instructor will probably not hold you responsible for proofs such as those in Examples 3 and 4, but ask to be sure.

6. "Wrong-way" integrals. There is no need to memorize these formulas. They can easily be recalled by using the fundamental theorem of calculus. Notice that $\int_a^a f(x)dx = F(a) - F(a) = 0$.

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7. Alternative version of the fundamental theorem. This version is not as important as $\int_a^b f(x)dx = F(b) - F(a)$ for this course. Note that differentiation is performed with respect to the upper endpoint. Use the concept of "wrong-way" integrals to differentiate with respect to the lower endpoint. For differentiation with respect to the lower limit, you can write $\int_x^b f(s)ds = -\int_b^x f(s)ds$.

8. Geometric interpretation of the fundamental theorem. Fig. 4.5.3 should help explain why the alternative version of the fundamental theorem is true, but the most important immediate goal is to understand the statement in the box on p. 237 .

9. Generalization of the fundamental theorem. Use the chain rule and the fundamental theorem of calculus to derive $(d/dt)\int_a^{g(t)} f(s)ds = f(g(t)) \times g'(t)$. Notice that if $g(t) = t$, the original version results. See Exercise 43.

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. Use the indefinite integral test. $(d/dx)(x^5 + C) = 5x^4$, which is the integrand, so the formula is correct.

5. (a) Use the indefinite integral test with the quotient rule.
$(d/dt)[t^3/(1 + t^3) + C] = [3t^2(1 + t^2) - t^3(3t^2)]/(1 + t^3)^2 =$
$3t^2/(1 + t^3)^2$, which is the integrand, so the formula is correct.

(b) Apply the fundamental theorem of calculus to get $\int_0^1 [3t^2/(1 + t^3)^2]dt =$
$[t^3/(1 + t^3)]\Big|_0^1 = 1/2$.

9. By the fundamental theorem of calculus, $\int_{-2}^3 (x^4 + 5x^2 + 2x + 1)dx =$
$(x^5/5 + 5x^3/3 + x^2 + x)\Big|_{-2}^3 = (243 + 32)/5 + 5(27 + 8)/3 + (9 - 4) +$
$(3 + 2) = 370/3$.

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13. Divide first to get $\int_1^2[(x^2 + 2x + 2)/x^4]dx = \int_1^2(x^{-2} + 2x^{-3} + 2x^{-4})dx = (-x^{-1} - x^{-2} - 2x^{-3}/3)\Big|_1^2 = (-1/2 - 1/4 - 1/12) - (-1 - 1 - 2/3) = 11/6$.

17. Guess that an antiderivative is $(1 + 2t)^6 + C$. The indefinite integral test yields $(d/dx)[(1 + 2t)^6 + C] = 2(6)(1 + 2t)^5$, so the antiderivative must be $(1/12)(1 + 2t)^6 + C$. By the fundamental theorem, $\int_1^2(1 + 2t)^5dt = (1/12)(1 + 2t)^6\Big|_1^2 = (1/12)(5^6 - 3^6) = (1/12)(15625 - 729) = 3724/3$.

21. Using Property 3 of integration, we get $\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = 3 + 4 = 7$.

25. This is a wrong-way integral. $\int_3^2 xdx = (x^2/2)\Big|_3^2 = (4 - 9)/2 = -5/2$.

29. The left-hand side is $(d/dx)\int_a^x(s^3 - 1)ds = (d/dx)[(s^4/4 - s)\Big|_a^x] = (d/dx)(x^4/4 - x - a^4/4 + a) = x^3 - 1 = f(x)$ because a is constant.

33. According to the alternative version of the fundamental theorem, $(d/dt)\int_a^t f(x)dx = f(t)$. When $f(x) = 3/(x^4 + x^3 + 1)^6$, the answer is $3/(t^4 + t^3 + 1)^6$.

37. Differentiating the distance function, $\int_a^t v(s)ds$, with respect to time yields the velocity function.

41. (a) Let $F(s)$ be an antiderivative for $f(s)$, then $F_1(t) = F(s)\Big|_{a_1}^t$ and $F_2(t) = F(s)\Big|_{a_2}^t$. $F_1(t) - F_2(t) = [F(t) - F(a_1)] - [F(t) - F(a_2)] = F(a_2) - F(a_1)$. Since a_1 and a_2 are constants, $F(a_2) - F(a_1)$ must be a constant.

(b) From part (a), $F_2(t) - F_1(t) = F(a_1) - F(a_2)$. By the fundamental theorem of calculus, this is $\int_{a_2}^{a_1} f(s)ds$.

45. The generalized chain rule was derived in Exercise 43. It states that $(d/dx)\int_a^{g(x)} f(t)dt = f(g(x))\cdot g'(x)$. Here, $g(x) = x^2$, $a = 1$, and $f(t) = 1/t$. Therefore, $F'(x) = (1/x^2)2x = 2/x$.

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49. Let $I = \int_a^b f(t)dt + \int_b^c f(t)dt$. We will show that every number less than I is a lower sum for f on $[a,c]$ and every number greater than I is an upper sum for f on $[a,c]$; by the definition of the integral, f will be integrable on $[a,c]$ and its integral will be I .

So let $S < I$. To show that S is a lower sum, we begin by writing $S = S_1 + S_2$, where $S_1 < \int_a^b f(t)dt$. and $S_2 < \int_b^c f(t)dt$. (See the hint on p. A.29.) This means that S_1 and S_2 are lower sums for f on $[a,b]$ and $[b,c]$, respectively. Thus there is a step function g_1 on $[a,b]$ with $g_1(t) \leqslant f(t)$ for all t in (a,b) , and $\int_a^b g_1(t)dt = S_1$, and there is a step function g_2 on $[b,c]$ with $g_2(t) \leqslant f(t)$ for all t in (b,c) and $\int_b^c g_2(t) = S_2$. Put together g_1 and g_2 to obtain a function g on $[a,c]$ by the definition:

$$g(t) = \begin{cases} g_1(t) & a \leqslant t < b \\ f(b) & t = b \\ g_2(t) & b < t \leqslant c \end{cases}.$$

The function g is a step function on $[a,c]$, with $g(t) \leqslant f(t)$ for all t in (a,c) . The sum which represents the integral for g on $[a,c]$ is the sum of the sums representing the integrals of g_1 and g_2 , so we have

$$\int_a^c g(t)dt = \int_a^b g_1(t)dt + \int_b^c g_2(t)dt = S_1 + S_2 = S$$

and S is a lower sum. Similarly, any number greater than I is an upper sum, so I is the integral of f on $[a,b]$.

Remark. Using the definition of "wrong-way integrals," one may easily verify that Property 3 holds no matter how a , b , and c are ordered.

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53. Since f is continuous, we can take the limit of the expression in Exercise 51. Using the definition of the derivative, we have $F'(t) = \lim_{h\to 0}\{[F(t+h) - F(t)]/h\} = \lim_{h\to 0}[(1/h)\int_t^{t+h} f(s)ds]$. From Exercise 52, the limit on the right equals $f(c)$ for $t \leq c \leq t+h$, but as $h \to 0$, $c \to t$. Therefore, the limit on the right is $f(t)$ as we wished to show.

SECTION QUIZ

1. Check the following integration formulas:
   (a) $\int(5x+3)(x+1)^3dx = (x+1/2)(x+1)^4 + C$.
   (b) $\int(3+4x)^7dx = (3+4x)^8/8 + C$.
   (c) $\int(x^4 + 2x^2)dx = 4(x^3 + x) + C$
2. Compute $\int_0^2 2xdx$, and $\left[\int_0^2 2dx\right]\cdot\left[\int_0^2 xdx\right]$. What do your results imply about the product of integrals?
3. Simplify the following expression to a single integral: $\int_0^3 f(x)dx + \int_3^7 f(x)dx - \int_7^0 f(x)dx$.
4. Let $G(t) = \int_{-t}^{t^2}(x+3)^3dx$. What is $G'(1)$?
5. A man buys some beachfront property for his alligator farm. If he puts a fence at the x-axis, the shoreline is given by $f(x) = x^3/3 - x + 1$ at high tide. He wants to know how much sand his alligators have to crawl around in.
   (a) Find the area for $-1 \leq x \leq 1$.
   (b) He decides to buy more land from $x = 1$ to $x = 2$. Use Property 3 of integration to compute the area for $-1 \leq x \leq 2$.
   (c) At low tide, the shoreline has the shape $2f(x)$. Find the area on $[-1,2]$.
   (d) How fast is the area changing (with respect to x) at $x = 1$?

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ANSWERS TO PREREQUISITE QUIZ

1. (a) $2x + 3$

   (b) $(5x^4 + 1)(x^2 + 3x + 1) + (x^5 + x)(2x + 3)$

   (c) $(-x^2 - 6x + 1)/(x^2 + 1)^2$

   (d) $(2x - 10)(x + 10)^2/x^2$

2. (a) $\sum_{i=1}^{25} i^2$

   (b) 6

   (c) $\sum_{k=6}^{10} k$

3. 64

4. $\int_a^b f(x)dx = F(x)\Big|_a^b$ where $F'(x) = f(x)$.

ANSWERS TO SECTION QUIZ

1. (a) The formula is correct.

   (b) The denominator should be 32 rather than 8 .

   (c) The right-hand side is the derivative when $C = 0$, not the integral.

2. $\int_0^2 2xdx = 4$, $(\int_0^2 2dx)(\int_0^2 xdx) = (4)(2) = 8$. In general, the product of two integrals does not equal the integral of the product of the integrands, i.e., $\int_a^b f(x)g(x)dx \neq [\int_a^b f(x)dx] \cdot [\int_a^b g(x)dx]$.

3. $2\int_0^7 f(x)dx$

4. $G'(t) = (t^2 + 3)^3(2t) + (-t + 3)^3$, so $G'(1) = 136$.

5. (a) 2

   (b) $\int_{-1}^2 f(x)dx = \int_{-1}^1 f(x)dx + \int_1^2 f(x)dx = 2 + 3/4 = 11/4$.

   (c) $\int_{-1}^2 2f(x)dx = 2(11/4) = 11/2$.

   (d) 1/3

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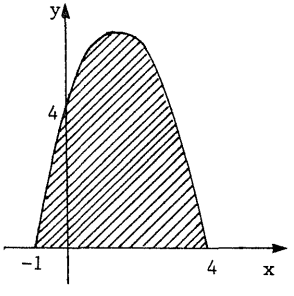
4.6 Applications of the Integral

PREREQUISITES

1. Recall how to calculate the area between a positive function and the x-axis (Sections 4.2 and 4.4).

PREREQUISITE QUIZ

1. Use integration to compute the area under $y = 3$ on $[-1,2]$.
2. Find the area under the curve $y = (x^3 + 1)/x^2$ for $1 \leqslant x \leqslant 2$.
3. 

Calculate the area of the shaded region, which is bounded by $y = -x^2 + 3x + 4$.

GOALS

1. Be able to compute the area of a region between two curves.
2. Be able to compute the total change in a quantity from its rate of change.

STUDY HINTS

1. Area versus signed area. Remember that the integral gives the signed area; therefore, the area of regions where the function is negative is the negative of the integral for the same region, i.e., $-\int_a^b f(x)dx$ wherever $f(x) < 0$.

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2. Area between curves. You should memorize the formula. Always subtract the lower curve from the higher curve. The formula works whether or not the curves lie above the x-axis. Note that if $f(x) = 0$, we get the formula for the area under the positive function, $g(x)$.

3. Infinitesimal approach. Try to understand this approach for deriving the area formula. Learning this method will help you derive formulas which will be introduced in future chapters as well as reduce the risk of memory lapses during exams.

4. Intersecting curves. It is a good idea to sketch all regions in area problems to check for points of intersection. These points must be known to determine the endpoints of integration. Again, in the area formula, subtract the lower curve from the higher one on each interval.

5. Multi-functional boundaries. If an area is bounded by several functions, it must be divided into subregions which are bounded by only two functions. Then, apply the additive property of areas.

6. Symmetry. Noting some symmetry can simplify a problem. For example, the area under $y = x^2$ on $[-1,1]$ is $2\int_0^1 x^2 dx$.

7. Integrating rates. Know that integrating rates of change gives the total change. Your instructor may choose to emphasize some examples over others, but only the terminology differs. For example, the rate, $r(x)$, may be called velocity, marginal revenue, current, productivity, or many other names depending on the application.

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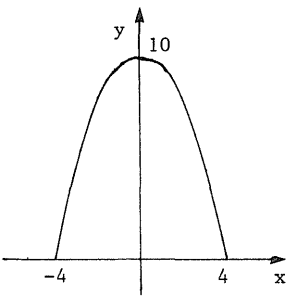
SOLUTIONS TO EVERY OTHER ODD EXERCISE

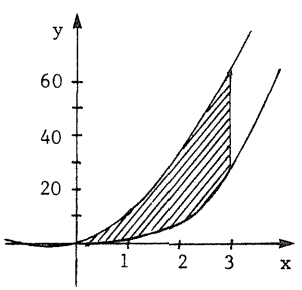
1. 

Place the parabolic arch as shown at the left. The curve has the equation $y = ax^2 + b$. When $x = 0$, we get $b = 10$. At the x-axis, we have $0 = 16a + 10$, so $a = -5/8$. Thus, the enclosed area is $\int_{-4}^{4}(-5x^2/8 + 10)dx = (-5x^3/24 + 10x)\Big|_{-4}^{4} = -80/3 + 80 = 160/3$.

5. $x^2/4 - 1 = 1 - x^2/4$ implies $x = \pm 2$ are the intersection points of the two curves. The area is $\int_{-2}^{2}[(1 - x^2/4) - (x^2/4 - 1)]dx = \int_{-2}^{2}(2 - x^2/2)dx = (2x - x^3/6)\Big|_{-2}^{2} = 8 - 8/3 = 16/3$.

9. On the interval $[0,1]$, $\sqrt{x} \geq x$, so the area is $\int_0^1(x^{1/2} - x)dx = (2x^{3/2}/3 - x^2/2)\Big|_0^1 = 1/6$.

13. $3x^2 \leq 3/4$ in $[-1/2,1/2]$ while $x^4 + 2 \geq 2$ on the same interval, so the area is $\int_{-1/2}^{1/2}(x^4 + 2 - 3x^2)dx = (x^5/5 + 2x - x^3)\Big|_{-1/2}^{1/2} = (1/160 + 1 - 1/18) - (-1/160 - 1 + 1/18) = 141/80$.

17. 

The area of the region sketched at the left is $\int_0^3(5x^2 + 6x - x^3)dx = (5x^3/3 + 3x^2 - x^4/4)\Big|_0^3 = 45 + 27 - 81/4 = 207/4$.

21. Use Method 2 of Example 8. $y^2 - 3 = 2y$ gives the limits of integration as 3 and -1. Since $2y \geq y^2 - 3$ on $[-1,3]$, the area is $\int_{-1}^{3}[2y - (y^2 - 3)]dy = (y^2 - y^3/3 + 3y)\Big|_{-1}^{3} = 8 - 28/3 + 12 = 32/3$.

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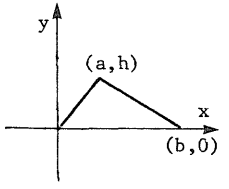
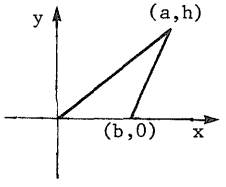
25. Use the formula $\Delta Q = Q(b) - Q(a) = \int_a^b r(t)dt$. Here, $r(t) = 300t^2$, $a = 0$, and $b = 5$. Therefore, the number of liters released is $\int_0^5 300t^2 dt = 100t^3|_0^5 = 100(125) = 12{,}500$ liters.

29. (a)



The equation of the line segment from $(0,0)$ to (a,h) is $y = (h/a)x$. For the segment from (a,h) to $(b,0)$, it is $y = [-h/(b - a)](x - b)$. The area is $\int_0^a (h/a)x\,dx + \int_a^b [-h/(b - a)](x - b)dx = (h/a)x^2/2|_0^a + [-h/(b - a)]\,[x^2/2 - bx]\,|_a^b = ha/2 + (-h)[(b - a)/2 - b] = hb/2$. Geometrically, the height is h and the base is b ; therefore, the area is $(1/2)(\text{base})(\text{height}) = bh/2$. Thus, both methods give the same answer.

(b)



Again, the equations are $y = (h/a)x$ and $y = [-h/(b - a)](x - b)$. The area is $\int_0^b (h/a)x\,dx + \int_b^a [(h/a)x - ((-h)/(b - a)) \times (x - b)]\,dx = (h/a)(x^2/2)|_0^b + (h/a)(x^2/2)|_b^a + [h/(b - a)]\,(x^2/2 - bx)|_b^a = hb^2/2a + ha/2 - hb^2/2a + (-h)[(a + b)/2 - b] = bh/2$. Geometrically, the height is h and the base is b , so the area is $bh/2$.

33. The equation of a parabolic curve is $y = cx^2$. Put the corner of the parabola at the origin, then the center of the roof is located at $(12,40)$. Therefore, c must be $3/400$. The area of the wall with the exhaust fans is $(20)(80) + 2\int_0^{40} (3x^2/400)dx = 1600 + (3/200)(x^3/3)|_0^{40} = 1600 + 320 = 1920$, so the volume is $(1920)(180) = 345{,}600$ cubic feet. The four fans can move $4(5500) = 22000$ cubic feet. The elapsed time is $345{,}600/22{,}000 = 15.71$ minutes or 15 minutes, 43 seconds.

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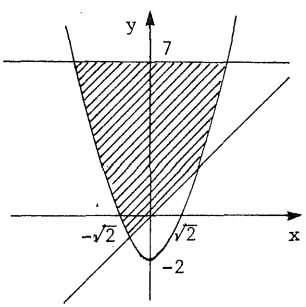
37. Let the horizontal line be drawn at $y = 1/c^2$, then the area of the upper region is $\int_1^c (1/x^2 - 1/c^2)dx = (-x^{-1} - x/c^2)\Big|_1^c = -1/c - 1/c + 1 + 1/c^2 = 3/8$. Algebraic manipulations yield $5c^2/8 - 2c + 1 = 0$. The solution is $(8 + 2\sqrt{6})/5$ because $1 < c < 4$. $1/c = 5/(8 + 2\sqrt{6}) = (4 - \sqrt{6})/4$ by rationalizing the denominator. Finally, we square to get $1/c^2 = y = (11 - 4\sqrt{6})/8$.

SECTION QUIZ

1. 

Find the area of the shaded region which is bounded by the curves $y = 7$, $y = x^2 - 2$, and $y = x$.

2. Find the area between $y = x^2 - 4x$ and $y = -3$ on $[-1,4]$.

3. The Hi-Ho Mining Company produces coal. The head dwarf at the company determines that at a depth of x meters, they can get $6 + 2x$ tons of coal.

   (a) How much can be mined in the first 20 meters?

   (b) How much can be mined at a depth of exactly 50 meters?

4. Mr. and Mrs. Chip have just been blessed with a handsome cookie monster child. Since consumption of cookies causes expansion of cookie monsters' stomachs, the child can eat $(x^2/2 + 10)$ thousand cookies per year in the x^{th} year of life. How many cookies do Mr. and Mrs. Chip need to buy for their child between the ages of 7 and 14 ?

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ANSWERS TO PREREQUISITE QUIZ

1. $\int_{-1}^{2} 3dx = 9$

2. 2

3. $\int_{-1}^{4}(-x^2 + 3x + 4)dx = 125/6$

ANSWERS TO SECTION QUIZ

1. 63/2

2. 28/3

3. (a) 520 tons

   (b) 106 tons

4. $\int_{7}^{14}(x^2/2 + 10)dx = (2821/6)$ thousand cookies

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4.R Review Exercises for Chapter 4

SOLUTIONS TO EVERY OTHER ODD EXERCISE

1. $\sum_{i=1}^{4} i^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2 = 1 + 4 + 9 + 16 = 30$.

5. Using the properties of summation, $\sum_{i=1}^{500} (3i + 7) = 3 \sum_{i=1}^{500} i + \sum_{i=1}^{500} 7$. Then, applying the formula $\sum_{i=1}^{n} i = n(n + 1)/2$ gives $3(500)(501)/2 + 7(500) = 375{,}750 + 3500 = 379{,}250$.

9. This is a step function. Using the partition $(0,1/5,1/4,1/3,1/2,1)$, we have $\int_0^1 f(x)dx = 1(1/5 - 0) + 2(1/4 - 1/5) + 3(1/3 - 1/4) + 4(1/2 - 1/3) + 5(1 - 1/2) = 1/5 + 1/10 + 1/4 + 2/3 + 5/2 = 223/60$.

13. Using the fundamental theorem of calculus, $\int_3^5 (-2x^3 + x^2)dx = (-x^4/2 + x^3/3)\Big|_3^5 = (-625 + 81)/2 + (125 - 27)/3 = -272 + 98/3 = -718/3$.

17. On the interval $[0,1]$, $x^3 + x^2 \geqslant 0$, so the area under the curve is $\int_0^1 (x^3 + x^2)dx = (x^4/4 + x^3/3)\Big|_0^1 = 1/4 + 1/3 = 7/12$.

21. (a) If intervals of equal length are used, then at least 10 intervals are required to satisfy the condition that the upper and lower sums are within 0.2 of one another. In this case, the upper sum is $\sum_{i=0}^{9} [4/(1 + (i/10)^2)](1/10) = 3.2399$. The lower sum is $\sum_{i=1}^{10} [4/(1 + (i/10)^2)](1/10) = 3.0399$.

(b) The average of 3.2399 and 3.0399 is 3.1399 , so we guess that the exact value of the integral is approximately 3.1399 . (In fact, the actual value of the integral is π .)

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25. Using calculus, the area is $\int_{a_1}^{a_2}(mx + b)dx = (mx^2/2 + bx)\Big|_{a_1}^{a_2} = m(a_2^2 - a_1^2)/2 + b(a_2 - a_1) = [(a_2 - a_1)/2](ma_1 + b + ma_2 + b)$. Alternatively, the area under the curve is a trapezoid. Plane geometry tells us that $A = (h/2)(b_1 + b_2) = [(a_2 - a_1)/2][(ma_1 + b) + (ma_2 + b)]$, which is exactly the same. (b_1 is $f(a_1)$ and b_2 is $f(a_2)$.)

29. (a) By the indefinite integral test on p. 233, we must differentiate the right-hand side and see if we get the integrand $x^2/(x^3 + 6)^2$. By the quotient rule, the derivative of the right-hand side is $(1/12)[(3x^2)(x^3 + 6) - (x^3 + 2)(3x^2)]/(x^3 + 6)^2 = (1/12)(12x^2)/(x^3 + 6)^2 = x^2/(x^3 + 6)^2$; therefore, the formula is verified by the indefinite integral test.

(b) The area is $\int_0^2[x^2/(x^3 + 6)^2]dx = (1/12)[(x^3 + 2)/(x^3 + 6)]\Big|_0^2 = (1/12)(10/14 - 2/6) = (1/12)(8/21) = 2/63$.

33. This is best done by integrating in y . $y^2 - 6 = y$ implies that the limits of integration are -2 and 3 . On $[-2,3]$, $y \geqslant y^2 - 6$, so the area is $\int_{-2}^{3}[y - (y^2 - 6)]dy = (y^2/2 - y^3/3 + 6y)\Big|_{-2}^{3} = 5/2 - 35/3 + 30 = 125/6$.

37. In each case, let $L(t)$ be the amount of leakage. Then as long as the tank does not empty for more than an instant, the volume of water at the end of 3 minutes is given by $1 + \int_0^3(3t^2 - 2t + 3 - L)dt$. If the tank does empty, then the volume is given by $\int_x^3(3t^2 - 2t - 3 - L)dt$, where x is the time the tank begins to fill again.

(a) Here, the rate of volume increase is $3t^2 - 2t + 1$, which is always positive, so $V(3) = 1 + \int_0^3(3t^2 - 2t + 1)dt = 1 + (t^3 - t^2 + t)\Big|_0^3 = 1 + 27 - 9 + 3 = 22$ liters.

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37. (b) The rate of volume increase is $3t^2 - 2t - 1$, which is ≤ 0 for $-1/3 \leq t \leq 1$. However, $1 + \int_0^x (3t^2 - 2t - 1)dt = 1 + x^3 - x^2 - x = x^2(x - 1) + (-1)(x - 1) = (x^2 - 1)(x - 1)$. This shows that volume is 0 only at $t = 1$ minute, so $V(3) = 1 + (t^3 - t^2 - t)\Big|_0^3 = 1 + 27 - 9 - 3 = 16$ liters.

(c) The rate of volume increase is $3t^2 - 2t - 5$, which is ≤ 0 for $-1 \leq t \leq 5/3$. If we use $V(x) = 1 + \int_0^x (3t^2 - 2t - 5)dt$, then $V(1) = -4$, so the tank must be empty for more than an instant. Therefore, $V(3) = \int_{5/3}^3 (3t^2 - 2t - 5)dt = (t^3 - t^2 - 5t)\Big|_{5/3}^3 = 3 - (-175/27) = (256/27)$ liters.

41. The distance travelled is $\int_a^b |v(t)|dt$. Velocity is 0 at $t = -1$ and 5, which is where velocity changes direction. Thus, the distance travelled is $-\int_0^5 (t^2 - 4t - 5)dt + \int_5^6 (t^2 - 4t - 5)dt = -(t^3/3 - 2t^2 - 5t)\Big|_0^5 + (t^3/3 - 2t^2 - 5t)\Big|_5^6 = 100/3 + 10/3 = 110/3$.

45. (a) $C(t) = \int (1000t - 7000)dt = 500t^2 - 7000t + K$. $C(0) = 40{,}000$ implies $K = 40{,}000$. We want to solve $C(t) = 20{,}000$, so $20{,}000 = 500t^2 - 7000t + 40{,}000$ implies $0 = 500(t^2 - 14t + 40) = 500(t - 4)(t - 10)$. Due to the restriction that $0 \leq t \leq 6$, the only solution is $t = 4$. Thus after 4 days, the concentration has dropped to half its original value, so the inspector should be sent on the fifth day.

(b) The total change in concentration from the fourth to the sixth day is $\int_4^6 1000(t - 7)dt = 1000\int_4^6 (t - 7)dt = 1000(t^2/2 - 7t)\Big|_4^6 = 1000(10 - 14) = -4000$. Therefore, the concentration drops by 4000 bacteria per cubic centimeter.

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49. By the alternative version of the fundamental theorem of calculus, $(d/dx)\int_0^x [s^2/(1 + s^3)]\,ds = x^2/(1 + x^3)$.

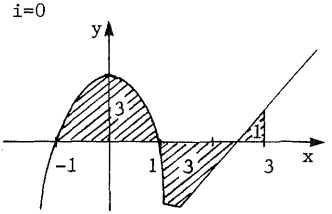
53. Apply the generalized version of the fundamental theorem, which states $(d/dt)\int_a^{g(t)} f(y)dy = f(g(t))\cdot g'(t)$. Here $g(t) = t^3 + 2$, $a = 0$, and $f(y) = 1/(y^2 + 1)$. Thus, the derivative is $3t^2/[(t^3 + 2)^2 + 1]$.

TEST FOR CHAPTER 4

1. True or false.

(a) $\sum_{i=0}^{10} 3 = 33$.

(b)



The areas of the shaded humps are 3 , 3 , and 1 , respectively; therefore, $\int_{-1}^{3} f(x)dx = 7$.

(c) If $F(x) = \int_7^x (dt/t)$, then $F'(x) = (1/t)\Big|_7^x$.

(d) $\sum_{i=1}^{10} 2^i + \sum_{k=1}^{19} (1/k)$ has no value because the dummy indices are different.

(e) $\int x^2(x^3 + 2)^4 dx = (x^3 + 2)^5/15 + C$.

2. Circle the correct conclusion.

(a) The fundamental theorem of calculus states that

(i) $\int_a^b f(x)dx = f'(b) - f'(a)$.

(ii) $\int_a^b f'(x)dx = f(b) - f(a)$.

(iii) $\int_a^b f(x)dx = F'(x)\Big|_a^b$ where $F'(x) = f(x)$.

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2. (b) The fundamental theorem of calculus states that

   (i) $(d/dx)\int_a^x f(t)dt = f(t)$.

   (ii) $(d/dx)\int_a^x f(t)dt = f(x)$.

   (iii) $(d/dx)\int_x^a f(t)dt = f(x)$.

   (iv) $(d/dx)\int_a^x f(t)dt = F(x)$ where $F'(x) = f(x)$.

3. Let $f(x) = (x^3 + 6)/(x^2 - x + 3)$; what is $\int_{-1}^{1} f'(u)du$?

4. Compute:

   (a) $\int_{60}^{50}\left(\sum_{i=1}^{19} i\right)dx$

   (b) $(d/dx)\int_{17}^{19}(t^5 + 4t^4 + 2t^2 + 4)dt$

   (c) $\sum_{i=5}^{48} [i/(i - 1) - (i - 2)/(i - 3)]$. Leave your answer as a sum of fractions.

   (d) $\int_0^1 [(x^2 + x - 6)/(x + 3)]dx$

5. Show that $\int_3^5 [(x + 5)/(x + 2)]dx \leqslant 16/5$.

6. There is only one region bounded by the ellipse $x^2/9 + y^2/4 = 1$ and $x = |1|$ which contains the origin. Express the area of this region as an integral, but don't evaluate it.

7. Find the area of the region bounded by $y = x$ for $x \geqslant 0$, $y = 5x$ for $x \leqslant 0$, and $6 - x^2 = y$ for all x .

8. (a) Differentiate $(2x + 3)^3(3x + 1)^2$.

   (b) Compute $\int_0^1 3(2x + 3)^2(3x + 1)(5x + 4)dx$.

9. Evaluate the following integrals:

   (a) $\int [(t^4 - 3t^2 + 1)/t^2]dt$

   (b) $\int_5^{10} f(x)dx + \int_{10}^{20} f(t)dt + \int_{20}^{5} f(s)ds$, where $f(x) = x^5 + 3 - 1/x^7$.

   (c) $\int_1^2 (t + 1)(t - 1)dt$

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10. Breakout Bob has just escaped into the West. The sheriff immediately mounts his horse and starts after Bob. The horse runs at $(20 - t^2)$ kilometers/hour. After 240 minutes, the sheriff lassos Bob off his horse. How far did the sheriff chase after Bob?

ANSWERS TO TEST FOR CHAPTER 4

1. (a) True
   (b) False; it is 1 .
   (c) False; $F'(x) = 1/x$, not $1/x - 1/7$.
   (d) False; the name given to an index does not affect the value of a sum.
   (e) True
2. (a) (ii)
   (b) (ii)
3. 4/3
4. (a) -1900
   (b) 0
   (c) 48/47 + 47/46 - 4/3 - 3/2
   (d) -3/2
5. $(x + 5)/(x + 2) \leq 8/5$ on $[4,5]$. Use Property 5 of integration.
6. $4\int_{-1}^{1}\sqrt{1 - x^2/9}\, dx = 8\int_{0}^{1}\sqrt{1 - x^2/9}\, dx$.
7. 184/3
8. (a) $6(2x + 3)^2(3x + 1)(5x + 4)$
   (b) 1973/2

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9. (a) $t^3/3 - 3t - 1/t + C$

   (b) 0

   (c) 4/3

10. (176/3) kilometers

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